Saturated tilings with dominoes and 2×2 squares

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Abstract: Two players play the following game on an $m \times n$ board. The first player occupies some disjoint 2×2 squares of the board. Then the second player tries to fill up the remaining space with 2×2 squares and special dominoes. Concentrating on the uncovered area, the first player wants to maximize, the second player wants to minimize. We show the somewhat unusual phenomenon that there is an equilibrium at $\frac{nm}{5}$. The second player can always reach this, whatever the first player selected in his turn. On the other hand, the first player has a construction leaving $\frac{nm}{5}$ empty squares such that the second player can not make any move.

Keywords: Island, board, tiling, domino

1 Introduction

The problem in our focus originates from a recently popular theme, called islands [1, 3]. Let a rectangular $m \times n$ board be given consisting of mn unit squares, called *cells*. Two cells are neighbors if they have a point in common. We associate a number (real or integer) to each cell of the board. We can think of this number as a height above see level. A subrectangle I of the board is called an *island* if and only if the height of each cell of I is greater than the height of any cell in the neighborhood of I. An island consisting of a single cell is called a *cellular* island. Other type of boards, islands and neighborhood can be considered. These objects or similar phenomena are naturally present in Information Theory, Algebraic Topology, Differential Geometry and probably elsewhere. Note for instance the proof [5] of Euler's theorem on polyhedra that uses local minima, saddles, and local maxima on a spherical surface [2].

The first observation is that two different islands are either disjoint or containing. This fact gives rise to a natural representation of the set of islands of a fixed board. Each island corresponds to a vertex of an auxiliary rooted tree (T, r). The root is the entire board and placed on the top of the figure. The vertex corresponding to island I_2 is the son of the vertex of I_1 if and only if $I_2 \subset I_1$ and there is no island between them.

The first question posed in this field was the maximum number of different islands of a rectangular $m \times n$ board [3], that is to maximize the number of vertices of (T, r) [1]. We give

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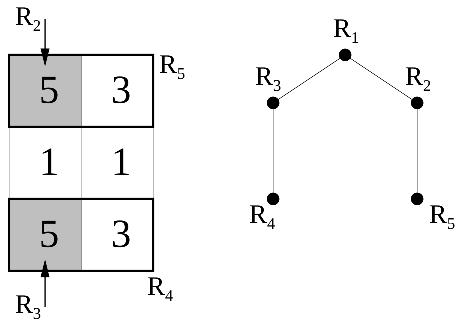

Figure 1: Hasse diagram of islands with respect to containment

an equivalent formulation of this problem using a dual grid, where the problem translates to a tiling with dominoes and quartets, that are 2×2 squares. The main result of [3] more or less follows from the fact that the area covered by dominoes and quartets is at most the area of the dual grid.

If we do not want to be specific, we use the word *tile* replacing domino or quartet.

At this point, it is tempting to ask some related questions. One of them can be formulated as a game. Assume two players play the following. The first player occupies some disjoint quartets of an $m \times n$ board, toroidal say. Then the second player tries to fill up the remaining space with some tiles. There is an extra condition on the dominoes. The side of length 2 of any domino must coincide with another side of length 2: either of a quartet or of a domino. We will call these dominoes congruent dominoes. We show that the first player can achieve that there are at most $\frac{nm}{5}$ tiles on the board. On the other hand, the second player can always reach this number of tiles.

The paper is organized as follows. There is a section containing the basic problems. Then there is a section with results, answering at least one question. Then the proofs are listed in a separate section.

2 The dual grid and basic questions

Let R be a rectangular or toroidal board of size $m \times n$. We associate to R a dual board B. In the rectangular case, the dual board B has size $(n+1) \times (m+1)$, and each vertex of a cell in R is a middle point of a cell in B. In the toroidal case, some cells of the rectangular dual grid coincide. Therefore, the size of B is also $m \times n$, and B is a translate of R such that middle points of R coincide with vertices of B.

Suppose that a height function is given. If a cell c is an island of R, then none of the eight neighbors of c can be a cellular island. That is, if we consider all cellular islands and enlarge them by half a unit in each direction, then the interiors of the resulting squares are still disjoint. These quartets appear naturally on the dual grid. On the other hand, assume that a cellular island c is contained in a 2×1 island. This enlargement can be seen on the dual grid as a quartet unified with a congruent domino. Notice, that any relation $I_1 \subset I_2$ can be seen as a series of domino placements with a possible modification of congruence. But we do not need it for our purposes.

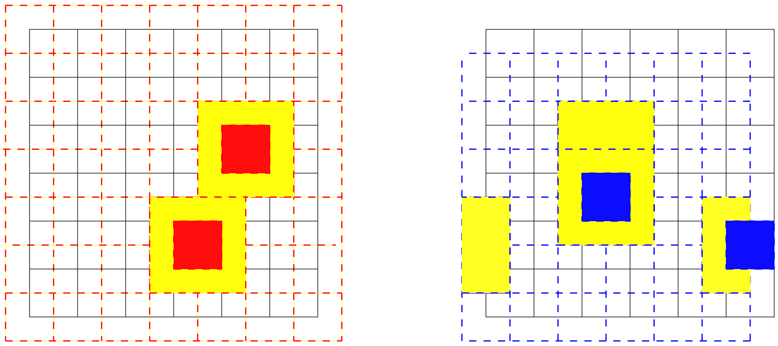

Figure 2: Rectangular and toroidal dual grid

It is known that the maximum number of islands on the rectangular board can be reached in several ways. In those optimal situations there might be just a few cellular islands or there can be rather many. Therefore, it is natural to seek an algorithm which extends a given set of cellular islands to a system of islands that is as large as possible. So far, we had the freedom to define the height of the cells, and the islands were given automatically. Now we talk about the islands only. It is clear that once we fix the islands, we can define a height function accordingly.

Question A: Assume there is a rectangular $m \times n$ board given. Let a set of cellular islands be A. What is the maximum cardinality of a system of rectangular islands containing A? Is there an algorithm which finds such a maximal system?

The set A corresponds to a partial tiling of the dual board B. There are easy geometric reasons, why such a partial tiling can not be extended into an optimal solution. The tiles form some closed regions of B. Whenever the cardinality of a region is odd, there is no way to fill it up with tiles of even size.

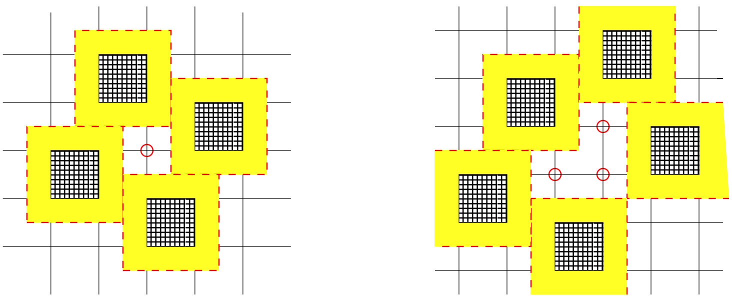

Figure 3: Obstacles to a complete covering

Question B: Assume there is a rectangular $m \times n$ board given. What are the maximal sets A of cellular islands that can be extended to a set of rectangular islands of maximum cardinality?

It is tempting to make the following conjecture: If the set A of cellular islands correspond to tiles of the dual grid such that each connected uncovered region has even size, then A is extendable to a maximum size system of islands. But this is false. A prototype counterexample can be seen in Figure 5.

We can also pose the question in the opposite way: the goal is to minimize the maximum cardinality. As it turns out, the sides of the rectangular board has some degeneracy. The cells on the side has less neighbors. Therefore, it is convenient to consider the toroidal board.

Question C: Assume there is a toroidal $m \times n$ board given. What are the best 'blocking' sets A of cellular islands, for which the maximum cardinality extensions are as small as possible?

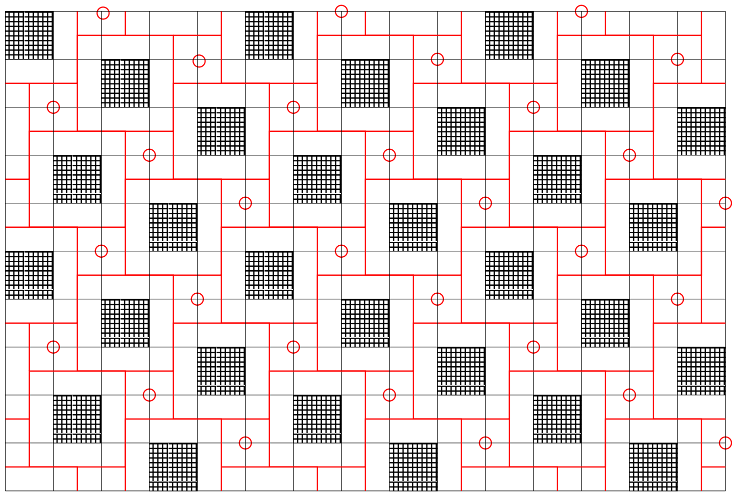

Figure 4: An unextendable tiling on the toroidal board

This problem is a relaxation of the mistiling ratio problem [6]. On the dual grid, A corresponds to a mistiling with quartets. The goal of a mistiling is to leave no space for another tile. In our case, the second player may use further tiles to cover some part of the remaining area, U. In this way, the first player try to maximize U by selecting A and the second player tries to minimize U.

Question D: Assume there is a toroidal $m \times n$ board given. What is the maximum cardinality M such that any set A of cellular islands can be extended to a set of rectangular islands of cardinality M?

This last problem is the main interest of us. The rest of the paper is devoted to give an answer.

3 Saturated tilings

Saturated structures belong to the core of Discrete Mathematics. For instance, a matching is called *saturated* if there is no way to add an edge to it keeping its matching property. The minimal size of a saturated matching is an important NP-hard graph parameter with numerous applications [9]. Since its introduction, there has been a continuous interest in the topic. See [4] for a recent result with an overview of the line of research.

Our setting is rather specific. Usually saturation means adding certain objects until there is no room for placing a new object. A maximal system of cellular islands is a saturated set of quartets in the dual grid, leading to the concept of mistiling.

The notion behind Question D is different. The cellular islands can spread. A minimal enlargement of a cellular island is extending a quartet in the dual grid by a congruent domino. Therefore, our saturation process is not just adding quartets until there is room for more, but includes another type of addition: placing a congruent domino next to a quartet.

Our main contribution is that we can determine the quantity M of Question D.

Theorem 1. *Let a toroidal $m \times n$ board be given. Assume that some quartets are already placed on the board. We can always extend this partial tiling with quartets and congruent dominoes such that the remaining empty area shrinks to at most $\frac{nm}{5}$.*

We notice that this bound is sharp in the following sense. Assume the cellular islands are placed on a $5k \times 5l$ board R such that they are a knight's move from each other, see Figure 4. Then the quartets on the dual board B leave no space for dominoes. It is easy to see that the uncovered area is $\frac{nm}{5}$.

Theorem 1 easily implies the following answer to Question D.

Theorem 2. *Any set A of cellular islands can be extended to a system of rectangular islands of size $\frac{nm}{5} + 1$ on the $m \times n$ toroidal board.*

To see this, assume that the number of quartets is N, the number of dominoes is D, and the number of empty squares is E. Counting the area yields $4N + 2D + E = nm$. If now $E \leq \frac{nm}{5}$, then $4N + 4D \geq 4N + 2D \geq \frac{4nm}{5}$. Since the number of vertices of the auxiliary rooted tree is at least $N + D + 1$, the statement follows.

Knowing the example in Figure 4, it is tempting to conjecture the strengthening of Theorem 2 that any set of cellular islands can be extended to a set of cellular islands of size $\frac{nm}{5}$. The following example shows the falsity of this claim.

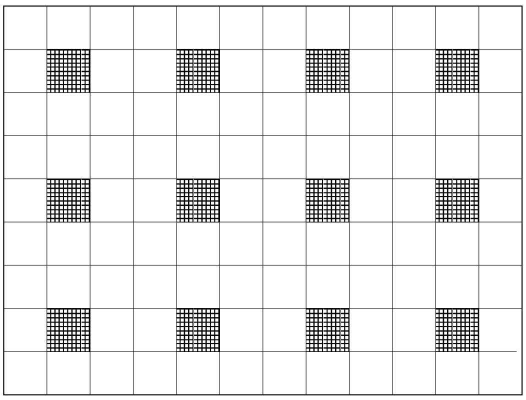

Figure 5: A 12×9 board with an unextendable set of 12 cellular islands

4 Solutions

Let us recall an exercise [7] from the Russian journal "KVANT". The solution of this problem inspired us to find a proof of our main result, Theorem 1. We did not know, but probably Sands [8] had been the first to prove this result.

Lemma 3. *Let an $n \times n$ board be given. We put some dominoes arbitrarily onto the board as long as we find an empty 2×1 subrectangle. Then the number of empty squares is at most $\frac{n^2}{3}$.*

Proof For simplicity, assume that the board is toroidal. Two squares of the board are neighbors if they share an edge. We know that no two empty squares are neighbors. Therefore, each empty square has precisely four dominoes as neighbors. On the other hand, any domino can have at most four empty squares as neighbors. Double counting gives us now that the number of empty squares is at most the number of dominoes. Therefore, the area of the empty part is at most one third.

If the board is planar not toroidal, then we have to distinguish between the squares with four or less neighbors. After some technical details we obtain the desired bound. □

Relying on the above idea, we are ready to prove our main result.

Proof [of Theorem 1] We can formulate the process as a game. The first (evil) player places a couple of quartets on the board B. Then we, the second player, try to fill the board as much as possible. We use the following greedy algorithm. If there is an empty 2×2 square, we occupy it with an extra quartet. After this subroutine is finished, we place congruent dominoes greedily as long as we can. The placement of dominoes induces the following changes to the tiles: if there is a (quartet, congruent domino) pair, we think of it as a 2×3 tile. Similarly, if there are two dominoes congruent to the same quartet, we see it as a 2×4 or a 3×3 tile etc. The largest tile we can produce is a 4×4 square. This step is also performed greedily. Therefore, the result is not a unique tiling. It is even possible that some dominoes remain unused. However, we claim that this two-step algorithm shrinks the empty area to at most $\frac{nm}{5}$.

Let an empty cell of the saturated board be called a *hole.* We would like to count the neighboring (hole, tile) pairs after the above filling-up algorithm. To do so, we need the following

Proposition 4. *Assume that the above filling-up algorithm terminated. If there are two holes forming a domino, then the middle point of the longer side of the domino is never a corner point of a tile.*

Let us postpone the proof for a moment. Assume the validity of the proposition and proceed with the proof of Theorem 1. The dual board B is partially covered by tiles. Let T_{ab} denote the number of tiles of size $a \times b$. Let the number of holes be E. We would like to count (hole, tile) neighbors. We create an auxiliary bipartite multigraph. The vertices in one class are the holes, and the other class consists of the tiles. We still have to define adjacency. Our strategy is that each hole should have valency 4. To achieve this, let a hole f and a tile t be adjacent if they have a side in common. They are also adjacent, if they have a side in common and there exists a hole e such that e and f are neighbors and e and t have a side in common. For instance, if there are three consecutive holes f_1, f_2, f_3 on the same side of tile t, then the contribution to the degree of f_1 is two, of f_2 is three and to t is seven. Practically, we count the (hole, hole) relations back and forth and transfer these two edges to the neighboring tile.

In this way a tile of size $a \times b$ has degree at most $6a + 6b - 20$, where $2 \leq a, b \leq 4$. Observe that each hole has degree four as we promised.

The total area of the dual grid is equal to the sum of the areas of the tiles plus the holes. To be more precise, the larger tiles contain some holes as well. This yields the following equation:

$$nm = 4T_{22} + 6T_{23} + 8T_{24} + 8T_{33} + 10T_{34} + 16T_{44} + E \qquad (1)$$

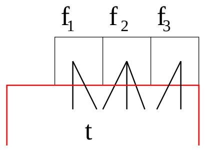

Figure 6: Adjacency between three holes and a tile

Remember that a tile of size $a \times b$ corresponds to $a+b-3$ nodes in the auxiliary rooted tree. Therefore, our goal is to prove that

$$\frac{nm}{5} \leq T_{22} + 2\,T_{23} + 3\,T_{24} + 3\,T_{33} + 4\,T_{34} + 5\,T_{44}.$$

Or equivalently

$$nm \leq 5\,T_{22} + 10\,T_{23} + 15\,T_{24} + 15\,T_{33} + 20\,T_{34} + 25\,T_{44}. \tag{2}$$

To achieve this, we have to bound the number of holes. Each hole has degree four and any tile of size $a \times b$ has degree at most $6a + 6b - 20$, where $2 \leq a, b \leq 4$. Therefore,

$$4E \leq 4\,T_{22} + 10\,T_{23} + 16\,T_{24} + 16\,T_{33} + 22\,T_{34} + 28\,T_{44}.$$

That is,

$$E \leq T_{22} + 2.5\,T_{23} + 4\,T_{24} + 4\,T_{33} + 5.5\,T_{34} + 7\,T_{44}.$$

Adding this to (1) yields

$$nm \leq 5\,T_{22} + 8.5\,T_{23} + 12\,T_{24} + 12\,T_{33} + 15.5\,T_{34} + 19\,T_{44},$$

which clearly implies (2). $\square$

This proof also shows that the empty part is maximal if we only use quartets. That is, we can answer Question C if we can determine the placement of quartets such that no congruent domino can be placed in the uncovered part of B. Applying Proposition 4 yields that the configuration in Figure 4 is the only solution.

Proof [of **Proposition 4**] Assume to the contrary that there is a vertical empty domino d, and the middle point of the right side coincides with the upper-left corner of a tile T_1. There might be some other holes below the domino.

If every left neighbor of tile T_1 is empty, then T_1 can either be extended to a larger tile, or there was an empty 2×2 area, in each case contradicting our algorithm. Therefore, the left boundary of T_1 ends lower than the series of holes below d. That is, there is another tile T_2 and its upper right corner is on the left boundary of T_1. Let f be the lowest hole neighboring T_1, on the left-hand side. There are two cases. Assume the square g on the left from f is empty. Then the square above g must be covered by a tile T_3, otherwise we found an empty 2×2 area, which is impossible. So there is a row of holes between T_2 and T_3. Therefore, one of the tiles T_2 and T_3 is extendable, a contradiction. So the square to the left from f is non-empty.

Let the upper empty square be f' in the original domino d. Then we are in a symmetric case, where f' play the role of f. If the rightnext square to f' say g' is empty, then there is a tile T_4

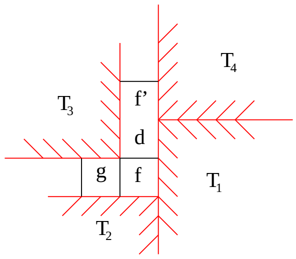

Figure 7: The holes should not leave space to a congruent domino

above g'. Now there is an empty row between T_1 and T_4, therefore one of the tiles is extendable. Otherwise, the left side of T_4 does not meet the right side of T_3, and therefore there is an empty column between T_3 and T_4. One of these tiles is extendable as before, a contradiction. □

Remark We strongly believe that the toroidal board can be changed into a rectangular board giving the same result with a more tedious proof requiring no extra ideas. The inclusion of such proof would make the presentation of the paper less clear giving no benefit to the reader and therefore left out. Note also that the toroidal case implies the asymptotic behavior of the rectangular board.

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